

Linear and nonlinear fractional Voigt models[★]

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Abstract. We consider fractional generalizations of the ordinary differential equation that governs the creep phenomenon. Precisely, two Caputo fractional Voigt models are considered: a rheological linear model and a nonlinear one. In the linear case, an explicit Volterra representation of the solution is found, involving the generalized Mittag-Leffler function in the kernel. For the nonlinear fractional Voigt model, an existence result is obtained through a fixed point theorem. A nonlinear example, illustrating the obtained existence result, is given.

Keywords: fractional differential equation, creep phenomenon, initial value problem, Mittag-Leffler function, fixed point theorem.

MSC 2010: 26A33, 34A08.

1 Introduction

To study the behaviour of viscoelastic materials, one often uses rheological models that can be of Voigt or Maxwell type or a combination of these basic models [23]. For example, the classical phenomenon of creep, in its simplest form, is known to be governed by a linear ordinary differential equation of order one, given by the linear Voigt model:

$$\eta \frac{d\epsilon(t)}{dt} + E\epsilon(t) = \sigma(t), \quad \sigma(0) = 0, \quad (1)$$

^{*} This is a preprint of a paper whose final and definite form will appear in the Springer LNEE book series. Submitted 16-April-2016; Revised 11-June-2016; Accepted 12-June-2016.

^{**} This work is part of first author's Ph.D., which is carried out at Houari Boumediene University, Algeria.

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where η is the viscosity coefficient and E is the modulus of the elasticity. For a given stress history σ , the solution of (1) is given by

$$\epsilon(t) = \frac{1}{\eta} \int_0^t e^{-\frac{t-s}{\tau}} \sigma(s) ds, \quad \tau = \frac{\eta}{E}, \quad (2)$$

where for $t \leq 0$ the material is at rest, without stress and strain. The constant τ is called the retardation time and has an analogous meaning to relaxation: it is an estimation of the time required for the creep process to approach completion. The expression

$$k(t) = \frac{1}{E} \left(1 - \exp \left(-\frac{t}{\tau} \right) \right), \quad t \geq 0, \quad (3)$$

is known as the creep function. In Section 2 we generalize (1)–(3).

Fractional calculus has recently become an important tool in the analysis of viscoelastic phenomena, such as stress-strain relationships in polymeric materials: in [17] the connection between the fractional calculus and the theory of Abel's integral equation is shown for materials with memory, while a fractional order Voigt model is proposed in [24] to better simulate the surface wave response of soft tissue-like material phantoms. For an historical survey of the contributions on the applications of fractional calculus in linear viscoelasticity, see [21]. In 1996, Mainardi investigated linear fractional relaxation-oscillation and fractional diffusion-wave phenomena [19]. Several other works in the same direction of research followed: for an introduction to the linear operators of fractional integration and fractional differentiation, accessible to applied scientists, we refer to [10]; for a comprehensive overview of fractional calculus and waves in linear viscoelastic media see [20]; for a book devoted to the description of the properties of the Mittag-Leffler function, its numerous generalizations and their applications in different areas of modern science, we refer to [9]; for a generalization of the partial differential equation of Gaussian diffusion, by using the time-fractional derivative, in both the Riemann–Liouville and Caputo senses, see [22]. Heymans and Podlubny have given a physical interpretation of initial conditions for fractional differential equations with Riemann–Liouville fractional derivatives [14]. Here, motivated by such results, we examine fractional creep equations involving Caputo derivatives of order $\alpha \in (0, 1)$. Caputo derivatives were chosen because they have a major utility for treating initial-value problems for physical and engineering applications, where initial conditions are usually expressed in terms of integer-order derivatives [1, 26]. Precisely, we begin by considering in Section 2 the following extension to (1):

$$\begin{cases} \eta^\alpha ({}^C D_0^\alpha \epsilon)(t) + E^\alpha \epsilon(t) = \sigma(t), & 0 < t \leq 1, \\ \epsilon(0) = 0, \end{cases} \quad (4)$$

where $E, \eta > 0$ and σ is a continuous function defined on $[0, 1]$. While the solution (2) of (1) is described by an exponential function, we show that the solution of (4) is expressed in terms of the Mittag-Leffler function (see Theorem 1), which

is a generalization of the exponential function and was introduced by Mittag-Leffler in [25], where he investigated some of their properties. The Mittag-Leffler function $E_\alpha(t)$ with $\alpha > 0$ is defined by the series representation

$$E_\alpha(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + 1)}, \quad \alpha > 0, \quad t \in \mathbb{C}, \quad (5)$$

where Γ denotes the Gamma function, valid in the whole complex plane. A straightforward generalization of the Mittag-Leffler function (5), due to Wiman [36] and used here, is obtained by replacing the additive constant 1 in the argument of the Gamma function in (5) by an arbitrary complex parameter β :

$$E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(\alpha n + \beta)}, \quad \alpha > 0, \quad \beta > 0, \quad t \in \mathbb{C}. \quad (6)$$

Mittag-Leffler functions are considered to be the queen functions of fractional calculus and they play a fundamental role in the solution to (4). Details about the Mittag-Leffler function and their importance when solving fractional differential equations can be found in [27, 33, 37] and references therein. Here we transform (4) as a Volterra integral equation to obtain an explicit solution involving the Mittag-Leffler function (see proof of Theorem 1). Moreover, we give a physical interpretation to the fractional order Voigt model (4) as a creep phenomenon, by finding the corresponding creep function (Theorem 2). Under some assumptions on σ , when it depends on ϵ (nonlinear Voigt model), in Section 3 we address the question of existence of positive solutions, which also contributes to the physical interpretation of the model (Theorem 3). Roughly speaking, the existence of nontrivial positive solutions is obtained by means of the Guo-Krasnosel'skii fixed point theorem. We end with an illustrative example and Section 4 of conclusions.

2 Solution to the fractional rheological linear Voigt model

Viscoelastic relations may be expressed in both integral and differential forms. Differential forms are related to rheological models and provide a more direct physical interpretation of the viscoelastic behavior. Integral forms are very general and appropriate for theoretical work. In Section 1 we introduced the fractional Voigt model (4) and explained its physical relevance. Here we make use of the corresponding integral representation to obtain an explicit solution to (4).

Theorem 1 (The fractional strain). *Assume that the given stress history σ of the fractional initial value problem (4) is a continuous function on $[0, 1]$. Then*

$$\epsilon(t) = \frac{1}{\eta^\alpha} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(- \left(\frac{t-s}{\tau} \right)^\alpha \right) \sigma(s) ds, \quad (7)$$

$0 \leq t \leq 1$, is the fractional strain, that is, is the solution to (4).

Proof. Since σ is a continuous function on $[0, 1]$, then we know from [16, Theorem 3.24] that the fractional initial problem (4) is equivalent to the Volterra integral equation of second kind

$$\epsilon(t) = \frac{1}{\eta^\alpha \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \sigma(s) ds - \frac{1}{\tau^\alpha \Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} \epsilon(s) ds.$$

To solve this integral equation, we apply the method of successive approximations. Let us consider the sequence defined by the following recurrence relation:

$$\epsilon_m = \frac{I^\alpha \sigma}{\eta^\alpha} - \frac{I^\alpha \epsilon_{m-1}}{\tau^\alpha}, \text{ where } I^\alpha z(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} z(s) ds. \text{ Setting } \epsilon_0 = \frac{I^\alpha \sigma}{\eta^\alpha},$$

we get $\epsilon_1 = \frac{I^\alpha \sigma}{\eta^\alpha} - \frac{I^{2\alpha} \sigma}{\eta^\alpha \tau^\alpha}$ and $\epsilon_2 = \frac{I^\alpha \sigma}{\eta^\alpha} - \frac{I^{2\alpha} \sigma}{\eta^\alpha \tau^\alpha} + \frac{I^{3\alpha} \sigma}{\eta^\alpha \tau^{2\alpha}}$. Continuing this process, we obtain that

$$\epsilon_m = \frac{1}{\eta^\alpha} \sum_{k=0}^m \left(-\frac{1}{\tau^\alpha} \right)^k I^{k\alpha+\alpha} \sigma.$$

Consequently, we have

$$\epsilon_m(t) = \frac{1}{\eta^\alpha} \int_0^t (t-s)^{\alpha-1} \sum_{k=0}^m \frac{(t-s)^{k\alpha}}{\Gamma(k\alpha+\alpha)} \left(-\frac{1}{\tau^\alpha} \right)^k \sigma(s) ds.$$

Taking the limit as $m \rightarrow \infty$, and by (6), we obtain the explicit solution (7). \square

Remark 1. Our fractional problem (4) provides a generalization to the linear Voigt creep model (1). If we take $\alpha = 1$, then Theorem 1 gives the solution (2) to the classical problem (1).

We now generalize the creep function (3) to our fractional Voigt model (4).

Theorem 2 (The fractional creep function). *The creep function associated with the fractional initial value problem (4) is given by*

$$k_\alpha(t) = - \left(\frac{\tau}{\eta} \right)^\alpha \left(E_\alpha \left(- \left(\frac{t}{\tau} \right)^\alpha \right) - 1 \right). \quad (8)$$

Proof. We find the creep function k_α by using (7), where the latter is defined as

$$\epsilon(t) = \int_0^t k_\alpha(t-s) d\sigma(s), \quad 0 \leq t \leq 1.$$

Integrating expression (7) by parts, we obtain that

$$\begin{aligned} \epsilon(t) &= \frac{1}{\eta^\alpha} t^\alpha E_{\alpha, \alpha+1} \left(- \left(\frac{t}{\tau} \right)^\alpha \right) \sigma(0) \\ &\quad + \frac{1}{\eta^\alpha} \int_0^t (t-s)^\alpha E_{\alpha, \alpha+1} \left(- \left(\frac{t-s}{\tau} \right)^\alpha \right) \sigma'(s) ds, \quad 0 \leq t \leq 1. \end{aligned}$$

The strain is linear in the stress. Therefore, the creep function is given by

$$k_\alpha(t) = \frac{1}{\eta^\alpha} t^\alpha E_{\alpha, \alpha+1} \left(- \left(\frac{t}{\tau} \right)^\alpha \right). \quad (9)$$

Now, by using the definition of Mittag-Leffler function in (9), we obtain that

$$\begin{aligned} k_\alpha(t) &= \frac{1}{\eta^\alpha} t^\alpha \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{t}{\tau}\right)^{\alpha n}}{\Gamma(\alpha n + \alpha + 1)} = \left(\frac{\tau}{\eta}\right)^\alpha \sum_{n=0}^{\infty} (-1)^n \frac{\left(\frac{t}{\tau}\right)^{\alpha n + \alpha}}{\Gamma(\alpha n + \alpha + 1)} \\ &= - \left(\frac{\tau}{\eta}\right)^\alpha \sum_{n=1}^{\infty} (-1)^n \frac{\left(\frac{t}{\tau}\right)^{\alpha n}}{\Gamma(\alpha n + 1)} = - \left(\frac{\tau}{\eta}\right)^\alpha \left(E_\alpha \left(- \left(\frac{t}{\tau} \right)^\alpha \right) - 1 \right). \end{aligned}$$

The proof is complete. \square

Remark 2. If we take $\alpha = 1$, then we obtain from Theorem 2 that

$$k_1(t) = \frac{\tau}{\eta} \left(1 - E_1 \left(- \left(\frac{t}{\tau} \right) \right) \right) = \frac{1}{E} \left(1 - \exp \left(- \left(\frac{t}{\tau} \right) \right) \right) = k(t),$$

that is, the creep function (3) is a special case of the $k_\alpha(t)$ given by (8).

Next we generalize (4) to the nonlinear case, where the stress σ depends on the strain ϵ .

3 A nonlinear fractional Voigt model

By applying the method of successive approximations, we have proved in Section 2 that the fractional initial value problem (4) has a solution ϵ in $C[0, 1]$ given by (7). Let us now consider (4) as a nonlinear problem, that is, consider a fractional Voigt model described by a differential equation with a nonlinear right-hand side σ depending on ϵ :

$$\begin{cases} \eta^\alpha ({}^C D_0^\alpha \epsilon)(t) + E^\alpha \epsilon(t) = \sigma(\epsilon(t)), & 0 < t \leq 1, \\ \epsilon(0) = 0. \end{cases} \quad (10)$$

We deal with the solvability of the initial value problem (10). Precisely, we are interested in proving the existence of positive solutions, which are the ones that make sense in physics. To establish existence of solutions has been a very active research area in mathematics. This is particularly true with respect to existence of solutions for fractional differential equations [2], which is also explained by the development of other fields of research, such as physics, mechanics and biology [1, 15, 30]. Many methods are used to prove existence of a solution, such as the fixed point technique, for which several theories are available [3, 4, 8]. In recent years, there has been many papers investigating the existence of positive solutions: see [7, 11, 18, 32, 34, 35] and references therein. In this section, motivated by many papers that discuss the existence of solutions to initial value problems, e.g. [5, 6, 12], we focus on the fractional initial value problem (10).

Theorem 3 (Existence of a positive solution to the nonlinear fractional Voigt model (10)). Assume that $\sigma : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a continuous, convex and decreasing function. Let $\mathbf{E}_0 := \lim_{\epsilon \rightarrow 0} \frac{\sigma(\epsilon)}{\epsilon}$ and $\mathbf{E}_\infty := \lim_{\epsilon \rightarrow \infty} \frac{\sigma(\epsilon)}{\epsilon}$. If $\mathbf{E}_0 = \infty$ and $\mathbf{E}_\infty = 0$, then problem (10) has at least one nontrivial positive bounded solution $\epsilon \in X$.

Our Theorem 3 is proved by the Guo–Krasnosel’skii fixed point theorem [13, 31]. Roughly speaking, our analysis is mainly based on the following result on the monotonicity of the Mittag-Leffler function, which was first proved by Schneider in [29]: the generalized Mittag-Leffler function $E_{\alpha,\beta}(-t)$ with $t \geq 0$ is completely monotonic if and only if $0 < \alpha \leq 1$ and $\beta \geq \alpha$. Thus, if $0 < \alpha \leq 1$ and $\beta \geq \alpha$, then $(-1)^n \frac{d^n}{dt^n} E_{\alpha,\beta}(-t) \geq 0$ for all $n = 0, 1, 2, \dots$. Note that $\sigma(0) \neq 0$ because our function σ is positive and decreasing and we are interested in a nontrivial solution.

3.1 Auxiliary results

Let $X = C[0, 1]$ be the Banach space of all continuous real functions defined on $[0, 1]$ with the norm $\|u\| = \sup_{t \in [0, 1]} |u(t)|$. Define the operator $T : X \rightarrow X$ by

$$T\epsilon(t) = \frac{1}{\eta^\alpha} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(- \left(\frac{t-s}{\tau} \right)^\alpha \right) \sigma(\epsilon(s)) ds, \quad 0 \leq t \leq 1. \quad (11)$$

Using the Guo–Krasnosel’skii fixed point theorem, we prove existence of nontrivial positive solutions. For that we first present and prove several lemmas. In what follows, K is the cone $K := \{\epsilon \in X : \epsilon(t) \geq 0, 0 \leq t \leq 1\}$.

Lemma 1. *The operator $T : K \rightarrow K$ is completely continuous.*

Proof. Taking into account the monotonicity of the Mittag-Leffler function, we have that the operator $T : K \rightarrow K$ is continuous in view of the assumptions of nonnegativeness and continuity of σ . Let $B \subset K$ be the bounded set $B := B(0, \eta_0) = \{\epsilon \in K : \|\epsilon\| \leq \eta_0, \eta_0 > 0\}$, and let $\rho = \max_{0 \leq t \leq 1, 0 \leq \epsilon \leq \eta_0} \sigma(\epsilon(t)) + 1$. Then, for any $\epsilon \in B$, we have

$$\begin{aligned} |T\epsilon(t)| &= \left| \frac{1}{\eta^\alpha} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(- \left(\frac{t-s}{\tau} \right)^\alpha \right) \sigma(\epsilon(s)) ds \right| \\ &\leq \frac{1}{\eta^\alpha} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(- \left(\frac{t-s}{\tau} \right)^\alpha \right) |\sigma(\epsilon(s))| ds \\ &\leq \frac{\rho}{\eta^\alpha \Gamma(\alpha+1)} t^\alpha \Rightarrow \|T\epsilon\| \leq \frac{\rho}{\eta^\alpha \Gamma(\alpha+1)}. \end{aligned}$$

Hence, $T(B)$ is uniformly bounded. Now, we prove that the operator T is equicontinuous for each $\epsilon \in B$, any $\varepsilon > 0$, and $t_1, t_2 \in [0, 1]$ with $t_2 > t_1$.

Let $\delta = \left(\frac{\eta^\alpha \Gamma(\alpha+1)\varepsilon}{2\rho} \right)^{\frac{1}{\alpha}}$. Then, for $|t_2 - t_1| < \delta$,

$$\begin{aligned} & |T\epsilon(t_1) - T\epsilon(t_2)| \\ & \leq \frac{\rho}{\eta^\alpha \Gamma(\alpha)} \left(\int_0^{t_1} ((t_1 - s)^{\alpha-1} - (t_2 - s)^{\alpha-1}) ds + \int_{t_1}^{t_2} (t_2 - s)^{\alpha-1} ds \right) \\ & \leq \frac{\rho((t_1^\alpha + (t_2 - t_1)^\alpha - t_2^\alpha + (t_2 - t_1)^\alpha)}{\eta^\alpha \Gamma(\alpha + 1)} \leq \frac{2\rho(t_2 - t_1)^\alpha}{\eta^\alpha \Gamma(\alpha + 1)} = \varepsilon. \end{aligned}$$

Therefore, $T(B)$ is equicontinuous. From the Arzela–Ascoli theorem, it follows that operator T is completely continuous. \square

The following results are also used in the proof of our Theorem 3.

Lemma 2 (Jensen’s inequality [28]). *Let μ be a positive measure and let Ω be a measurable set with $\mu(\Omega) = 1$. Let I be an interval and suppose that u is a real function in $L^1(\Omega)$ with $u(t) \in I$ for all $t \in \Omega$. If f is convex on I , then*

$$f\left(\int_{\Omega} u(t) d\mu(t)\right) \leq \int_{\Omega} (f \circ u)(t) d\mu(t).$$

Lemma 3 (Guo–Krasnosel’skii’s fixed point theorem [13]). *Let X be a Banach space and let $K \subset X$ be a cone. Assume Ω_1 and Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$, and let $T : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \rightarrow K$ be a completely continuous operator such that either*

1. $Tu \leq u$ for any $u \in K \cap \partial\Omega_1$ and $Tu \geq u$ for any $u \in K \cap \partial\Omega_2$, or
2. $Tu \geq u$ for any $u \in K \cap \partial\Omega_1$ and $Tu \leq u$ for any $u \in K \cap \partial\Omega_2$.

Then T has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

Since σ is continuous on \mathbb{R}_+ , we can define the function $\overline{\sigma}(\epsilon) = \max_{0 \leq z \leq \epsilon} \{\sigma(z)\}$.

Let $\overline{\mathbf{E}}_0 = \lim_{\epsilon \rightarrow 0} \frac{\overline{\sigma}(\epsilon)}{\epsilon}$ and $\overline{\mathbf{E}}_\infty = \lim_{\epsilon \rightarrow \infty} \frac{\overline{\sigma}(\epsilon)}{\epsilon}$.

Lemma 4 (See [34]). *Assume σ is continuous. Then $\overline{\mathbf{E}}_0 = \mathbf{E}_0$ and $\overline{\mathbf{E}}_\infty = \mathbf{E}_\infty$.*

We are now in condition to prove Theorem 3.

3.2 Proof of Theorem 3

By Lemma 1, we know that the operator (11) is completely continuous. Now, using Lemma 3, we give a proof to our result. Denote $\Omega_{r_i} = \{\epsilon \in X : \|\epsilon\| < r_i\}$. When $\mathbf{E}_0 = \infty$, we can choose $r_1 > 0$ sufficiently small such that $\sigma(\epsilon) \geq \varpi\epsilon$ for $\epsilon \leq r_1$, where ϖ satisfies $\left(\varpi \frac{E_{\alpha,\alpha}(-\frac{1}{\tau^\alpha})}{\eta^\alpha \alpha(\alpha+1)} \right) > 1$. Now let us show that $T\epsilon \leq \epsilon$

for any $\epsilon \in K \cap \partial\Omega_{r_1}$. In fact, if there exists $\epsilon_1 \in \partial\Omega_{r_1}$ such that $T\epsilon_1 \leq \epsilon_1$, the following inequalities hold:

$$\begin{aligned}
\|\epsilon_1\| &\geq \|T\epsilon_1\| \geq \int_0^1 T\epsilon_1(t) dt \\
&\geq \frac{1}{\eta^\alpha} \int_0^1 \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\left(\frac{t-s}{\tau}\right)^\alpha \right) \sigma(\epsilon_1(s)) ds dt \\
&\geq \frac{1}{\eta^\alpha} E_{\alpha,\alpha} \left(-\frac{1}{\tau^\alpha} \right) \int_0^1 \sigma(\epsilon_1(s)) \left(\int_s^1 (t-s)^{\alpha-1} dt \right) ds \\
&\geq \frac{E_{\alpha,\alpha} \left(-\frac{1}{\tau^\alpha} \right)}{\eta^\alpha \alpha} \int_0^1 (1-s)^\alpha \sigma(\epsilon_1(s)) ds \\
&\geq \frac{E_{\alpha,\alpha} \left(-\frac{1}{\tau^\alpha} \right)}{\eta^\alpha \alpha (\alpha+1)} \int_0^1 (\alpha+1)(1-s)^\alpha \sigma(\epsilon_1(s)) ds.
\end{aligned}$$

Then, by Lemma 2, we have

$$\begin{aligned}
\|\epsilon_1\| &\geq \frac{E_{\alpha,\alpha} \left(-\frac{1}{\tau^\alpha} \right)}{\eta^\alpha \alpha (\alpha+1)} \sigma \left(\int_0^1 (\alpha+1)(1-s)^\alpha \epsilon_1(s) ds \right) \\
&\geq \frac{E_{\alpha,\alpha} \left(-\frac{1}{\tau^\alpha} \right)}{\eta^\alpha \alpha (\alpha+1)} \sigma \left(\int_0^1 (\alpha+1)(1-s)^\alpha r_1 ds \right) \\
&\geq \frac{E_{\alpha,\alpha} \left(-\frac{1}{\tau^\alpha} \right)}{\eta^\alpha \alpha (\alpha+1)} \sigma(r_1) \geq \varpi \frac{E_{\alpha,\alpha} \left(-\frac{1}{\tau^\alpha} \right)}{\eta^\alpha \alpha (\alpha+1)} r_1 > r_1,
\end{aligned}$$

which is a contradiction. Since $\mathbf{E}_\infty = 0$, Lemma 4 implies $\bar{\mathbf{E}}_\infty = 0$. Thus, there exists $r_2 \in (r_1, \infty)$ such that $\bar{\sigma}(r_2) < \eta^\alpha \Gamma(\alpha+1)r_2$. Note that $0 < \Gamma(\alpha+1) < 1$ for all $\alpha \in (0, 1)$. We now show that $T\epsilon \geq \epsilon$ for any $\epsilon \in K \cap \partial\Omega_{r_2}$. If there exists $\epsilon_2 \in \partial\Omega_{r_2}$ such that $T\epsilon_2 \geq \epsilon_2$, then

$$\begin{aligned}
\|\epsilon_2\| &\leq \|T\epsilon_2\| = \sup_{t \in [0,1]} \frac{1}{\eta^\alpha} \int_0^t (t-s)^{\alpha-1} E_{\alpha,\alpha} \left(-\left(\frac{t-s}{\tau}\right)^\alpha \right) \sigma(\epsilon_2(s)) ds \\
&\leq \frac{1}{\eta^\alpha \Gamma(\alpha+1)} \max_{0 < \epsilon_2 < r_2} \sigma(\epsilon_2) \\
&\leq \frac{1}{\eta^\alpha \Gamma(\alpha+1)} \bar{\sigma}(r_2) < r_2,
\end{aligned}$$

which is a contradiction. Hence, from the first part of the Lemma 3, T has a fixed point in $K \cap (\bar{\Omega}_{r_2} \setminus \Omega_{r_1})$. Therefore, problem (10) has at least one nontrivial bounded positive solution $\epsilon \in X$.

3.3 An example

We now take a simple example to illustrate our analysis. Consider problem

$$\begin{cases} {}^C D_0^{\frac{1}{2}} \epsilon(t) + \sqrt{2} \epsilon(t) = \frac{1}{1+\epsilon(t)}, & 0 < t \leq 1, \\ \epsilon(0) = 0. \end{cases} \quad (12)$$

As already mentioned, the term $\frac{1}{1+\epsilon(t)}$ is the constitutive equation of the creep. Function $\sigma(\epsilon) = \frac{1}{1+\epsilon} : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is continuous, convex and decreasing with $\sigma(0) \neq 0$. Due to the fact that $\mathbf{E}_0 = \infty$ and $\mathbf{E}_\infty = 0$, it follows from Theorem 3 that (12) has at least one nontrivial bounded positive solution $\epsilon \in C[0, 1]$.

4 Conclusion

In this work we investigated the creep phenomenon described by linear and nonlinear fractional order Voigt models involving the Caputo derivative. We were able to give an integral representation of our initial value problem and to compute the creep function in the linear case. The obtained Volterra integral equation involves the Mittag-Leffler function in the kernel, which is a completely monotonic function in the context of our considerations. This property was the key of our analysis to establish existence of positive solutions.

Acknowledgments

This research was finished while Chidouh was visiting University of Aveiro, Portugal. The hospitality of the host institution and the financial support of Houari Boumedienne University, Algeria, are here gratefully acknowledged. Torres was supported by CIDMA and FCT within project UID/MAT/04106/2013. The authors are grateful to two referees for valuable comments and suggestions.

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